

Selfadjoint realization of boundary-value problems with interior singularities

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Abstract

The purpose of this paper is to investigate some spectral properties of Sturm-Liouville type problems with interior singularities. Some of the mathematical aspects necessary for developing own technique presented. By applying this technique we construct some special solutions of the homogeneous equation and present a formula and the existence conditions of Green's function. Further based on this results and introducing operator treatment in adequate Hilbert space we derive the resolvent operator and prove selfadjointness of the considered problem.

Keywords: Boundary-value problems, transmission conditions, Green function, resolvent operator, singular point.

1. Introduction

For inhomogeneous linear systems, the basic Superposition Principle says that the response to a combination of external forces is the self-same combination of responses to the individual forces. In a finite-dimensional system, any forcing function can be decomposed into a linear combination of unit impulse forces, each applied to a single component of the system, and so the full solution can be written as a linear combination of the solutions to the impulse problems. This simple idea will be adapted to boundary value problems governed by differential equations, where the response of the system to

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a concentrated impulse force is known as the Green's function. With the Green's function in hand, the solution to the inhomogeneous system with a general forcing function can be reconstructed by superimposing the effects of suitably scaled impulses. In this study we shall investigate some spectral properties of the Sturm-Liouville differential equation on two interval

$$\mathcal{L}y := -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [a, c) \cup (c, b] \quad (1)$$

on $[a, c) \cup (c, b]$, with eigenparameter- dependent boundary conditions at the end points $x = a$ and $x = b$

$$\tau_1(y) := \alpha_{10}y(a) + \alpha_{11}y'(a) = 0, \quad (2)$$

$$\tau_2(y) := \alpha_{20}y(b) - \alpha_{21}y'(b) + \lambda(\alpha'_{20}y(b) - \alpha'_{21}y'(b)) = 0 \quad (3)$$

and the transmission conditions at the singular interior point $x = c$

$$\tau_3(y) := \beta_{11}^- y'(c-) + \beta_{10}^- y(c-) + \beta_{11}^+ y'(c+) + \beta_{10}^+ y(c+) = 0, \quad (4)$$

$$\tau_4(y) := \beta_{21}^- y'(c-) + \beta_{20}^- y(c-) + \beta_{21}^+ y'(c+) + \beta_{20}^+ y(c+) = 0, \quad (5)$$

where the potential $q(x)$ is real continuous function in each of the intervals $[a, c)$ and $(c, b]$ and has a finite limits $q(c \mp 0)$, λ is a complex spectral parameter, α_{ij} , β_{ij}^\pm , ($i = 1, 2$ and $j = 0, 1$), α'_{ij} ($i = 2$ and $j = 0, 1$) are real numbers.

Our problem differs from the usual regular Sturm-Liouville problem in the sense that the eigenvalue parameter λ are contained in both differential equation and boundary conditions and two supplementary transmission conditions at one interior point are added to boundary conditions. Such problems are connected with discontinuous material properties, such as heat and mass transfer, vibrating string problems when the string loaded additionally with points masses, diffraction problems [14, 16] and varied assortment of physical transfer problems. We develop an own technique for investigation some spectral properties of this problem. In particular, we construct the Green function's and adequate Hilbert space for selfadjoint realization of the considered problem.

2. Some basic solutions and Green's function

Denote the determinant of the k-th and j-th columns of the matrix

$$T = \begin{bmatrix} \beta_{10}^- & \beta_{11}^- & \beta_{10}^+ & \beta_{11}^+ \\ \beta_{20}^- & \beta_{21}^- & \beta_{20}^+ & \beta_{21}^+ \end{bmatrix}$$

by Δ_{kj} . For selfadjoint realization in adequate Hilbert space, everywhere in below we shall assume that

$$\Delta_{12} > 0 \text{ and } \Delta_{34} > 0. \quad (6)$$

With a view to constructing the Green's function we shall define two special solution of the equation (1) by own technique as follows. At first consider the next initial-value problem on the left interval $[a, c)$

$$-y'' + q(x)y = \lambda y \quad (7)$$

$$y(a) = \alpha_{11}, \quad y'(a) = -\alpha_{10}. \quad (8)$$

It is known that this problem has an unique solution $u = \varphi^-(x, \lambda)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [a, c)$ (see, for example, [15]). By applying the similar method of [7] we can prove that the equation (1) on the right interval $(c, b]$ has an unique solution $u = \varphi^+(x, \lambda)$ satisfying the equalities

$$\varphi^+(c+, \lambda) = \frac{1}{\Delta_{12}}(\Delta_{23}\varphi^-(c-, \lambda) + \Delta_{24}\frac{\partial\varphi^-(c-, \lambda)}{\partial x}) \quad (9)$$

$$\frac{\partial\varphi^+(c+, \lambda)}{\partial x} = \frac{-1}{\Delta_{12}}(\Delta_{13}\varphi^-(c-, \lambda) + \Delta_{14}\frac{\partial\varphi^-(c-, \lambda)}{\partial x}). \quad (10)$$

which also is an entire function of the parameter λ for each fixed $x \in [c, b]$. Consequently, the solution $u = \varphi(x, \lambda)$ defined by

$$\varphi(x, \lambda) = \begin{cases} \varphi^-(x, \lambda), & x \in [a, c) \\ \varphi^+(x, \lambda), & x \in (c, b] \end{cases} \quad (11)$$

satisfies the equation (1) on whole $[a, c) \cup (c, b]$, the first boundary condition of (2) and both transmission conditions(4) and (5).

By the same technique, we can define the solution

$$\psi(x, \lambda) = \begin{cases} \psi^-(x, \lambda), & x \in [a, c) \\ \psi^+(x, \lambda), & x \in (c, b] \end{cases} \quad (12)$$

so that

$$\psi(b, \lambda) = \alpha_{21} + \lambda\alpha'_{21}, \quad \frac{\partial\psi(b, \lambda)}{\partial x} = \alpha_{20} + \lambda\alpha'_{20} \quad (13)$$

and

$$\psi^-(c-, \lambda) = \frac{-1}{\Delta_{34}}(\Delta_{14}\psi^+(c+, \lambda) + \Delta_{24}\frac{\partial\psi^+(c+, \lambda)}{\partial x}), \quad (14)$$

$$\frac{\partial\psi^+(c-, \lambda)}{\partial x} = \frac{1}{\Delta_{34}}(\Delta_{13}\psi^+(c+, \lambda) + \Delta_{23}\frac{\partial\psi^+(c+, \lambda)}{\partial x}). \quad (15)$$

Consequently, $\psi(x, \lambda)$ satisfies the equation (1) on whole $[a, c) \cup (c, b]$, the second boundary condition (3) and both transmission condition (4) and (5). By using (9), (10), (14) and (15) and the well-known fact that the Wronskians $W[\varphi^-(x, \lambda), \psi^-(x, \lambda)]$ and $W[\varphi^+(x, \lambda), \psi^+(x, \lambda)]$ are independent of variable x it is easy to show that $\Delta_{12}w^+(\lambda) = \Delta_{34}w^-(\lambda)$. We shall introduce the characteristic function for the problem (1) – (5) as

$$w(\lambda) := \Delta_{34}w^-(\lambda) = \Delta_{12}w^+(\lambda).$$

Similarly to [7] we can prove that, there are infinitely many eigenvalues λ_n , $n = 1, 2, \dots$ of the BVTP (1) – (5) which are coincide with the zeros of characteristic function $w(\lambda)$.

Now, let us consider the non-homogenous differential equation

$$y'' + (\lambda - q(x))y = f(x), \quad (16)$$

on $[a, c) \cup (c, b]$ together with the same boundary and transmission conditions (1) – (5), when $w(\lambda) \neq 0$. The following formula is obtained for the solution $Y = Y_0(x, \lambda)$ of the equation (16) under boundary and transmission conditions (2)-(5)

$$Y_0(x, \lambda) = \begin{cases} \frac{\Delta_{34}\psi^-(x, \lambda)}{\omega(\lambda)} \int_a^x \varphi^-(y, \lambda) f(y) dy + \frac{\Delta_{34}\varphi^-(x, \lambda)}{\omega(\lambda)} \int_x^{c-} \psi^-(y, \lambda) f(y) dy \\ \quad + \frac{\Delta_{12}\varphi^-(x, \lambda)}{w(\lambda)} \int_{c+}^b f(y) \psi^+(y, \lambda) dy, & \text{for } x \in [a, c) \\ \frac{\Delta_{12}\psi^+(x, \lambda)}{\omega(\lambda)} \int_{c+}^x \varphi^+(y, \lambda) f(y) dy + \frac{\Delta_{12}\varphi^+(x, \lambda)}{\omega(\lambda)} \int_x^b \psi^+(y, \lambda) f(y) dy \\ \quad + \frac{\Delta_{34}\psi^+(x, \lambda)}{w(\lambda)} \int_a^{c-} f(y) \varphi^-(y, \lambda) dy, & \text{for } x \in (c, b] \end{cases} \quad (17)$$

From this formula we find that the Green function's of the problem (1)-(5) has the form

$$G_0(x, y; \lambda) = \begin{cases} \frac{\varphi(y, \lambda)\psi(x, \lambda)}{\omega(\lambda)} & \text{for } a \leq y \leq x \leq b, \quad x, y \neq c \\ \frac{\varphi(x, \lambda)\psi(y, \lambda)}{\omega(\lambda)} & \text{for } a \leq x \leq y \leq b, \quad x, y \neq c \end{cases} \quad (18)$$

and the solution (17) can be rewritten in the terms of this Green function's as

$$Y_0(x, \lambda) = \Delta_{12} \int_a^{c-} G_0(x, y; \lambda) f(y) dy + \Delta_{34} \int_{c+}^b G_0(x, y; \lambda) f(y) dy. \quad (19)$$

3. Construction of the Resolvent operator by means of Green's function the in adequate Hilbert space

In this section we define a linear operator A in suitable Hilbert space such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. For this we assume that $\Delta_0 := \alpha_{21}\alpha'_{20} - \alpha_{20}\alpha'_{21} > 0$ and introduce a new inner product in the Hilbert space $H = (L_2[a, c) \oplus L_2(c, b]) \oplus \mathbb{C}$ by

$$\langle F, G \rangle_1 := \Delta_{12} \int_a^{c-} f(x) \overline{g(x)} dx + \Delta_{34} \int_{c+}^b f(x) \overline{g(x)} dx + \frac{\Delta_{34}}{\Delta_0} f_1 \overline{g_1}$$

for $F = (f(x), f_1)$, $G = (g(x), g_1) \in H$.

Remark 1. Note that this modified inner product is equivalent to standard inner product of $(L_2[a, c) \oplus L_2(c, b]) \oplus \mathbb{C}$, so $H_1 = (L_2[a, c) \oplus L_2(c, b]) \oplus \mathbb{C}$, $\langle \cdot, \cdot \rangle_1$ is also Hilbert space.

For convenience denote

$$T_b(f) := \alpha_{20}f(b) - \alpha_{21}f'(b), \quad T'_b(f) := \alpha'_{20}f(b) - \alpha'_{21}f'(b)$$

and define a linear operator

$$A(\mathcal{L}f(x), T'_b(f)) = (\mathcal{L}f, -T_b(f))$$

with the domain $D(A)$ consisting of all elements $(f(x), f_1) \in H_1$, such that $f(x)$ and $f'(x)$ are absolutely continuous in each interval $[a, c)$ and $(c, b]$, and has a finite limit $f(c \mp 0)$ and $f'_1(c \mp 0)$, $\mathcal{L}f \in L_2[a, b]$, $\tau_1 f = \tau_3 f = \tau_4 f = 0$ and $f_1 = T'_b(f)$.

Consequently the problem (1) – (5) can be written in the operator form as

$$AF = \lambda F, \quad F = (f(x), T'_b(f)) \in D(A)$$

in the Hilbert space H_1 . It is easy to see that, the operator A is well defined in H_1 . Let A be defined as above and let λ not be an eigenvalue of this operator. For construction the resolvent operator $R(\lambda, A) := (\lambda - A)^{-1}$ we shall solve the operator equation

$$(\lambda - A)Y = F \quad (20)$$

for $F \in H_1$. This operator equation is equivalent to the nonhomogeneous differential equation

$$y'' + (\lambda - q(x))y = f(x), \quad (21)$$

on $[a, c) \cup (c, b]$ subject to nonhomogeneous boundary conditions and homogeneous transmission conditions

$$\tau_1(y) = \tau_3(y) = \tau_4(y) = 0, \quad \tau_2(y) = -f_1 \quad (22)$$

Let $Im\lambda \neq 0$. Putting this general solution in (22) yields

$$d_{11} = \frac{\Delta_{12}}{\omega(\lambda)} \int_{c+}^b \psi^+(y, \lambda) f(y) dy + \frac{\Delta_{12} f_1}{\omega(\lambda)}, \quad d_{12} = 0 \quad (23)$$

$$d_{21} = \frac{\Delta_{12} f_1}{\omega(\lambda)}, \quad d_{22} = \frac{\Delta_{34}}{\omega(\lambda)} \int_a^{c-} \varphi^-(y, \lambda) f(y) dy \quad (24)$$

Thus the problem (21)-(22) has an unique solution,

$$Y(x, \lambda) = \begin{cases} \frac{\Delta_{34} \psi^-(x, \lambda)}{\omega(\lambda)} \int_a^x \varphi^-(y, \lambda) f(y) dy + \frac{\Delta_{34} \varphi^-(x, \lambda)}{\omega(\lambda)} \int_x^{c-} \psi^-(y, \lambda) f(y) dy \\ + \frac{\Delta_{12} \varphi^-(x, \lambda)}{\omega(\lambda)} \left(\int_{c+}^b \psi^+(y, \lambda) f(y) dy + f_1 \right), & \text{for } x \in [a, c) \\ \frac{\Delta_{12} \psi^+(x, \lambda)}{\omega(\lambda)} \int_{c+}^x \varphi^+(y, \lambda) f(y) dy + \frac{\Delta_{12} \varphi^+(x, \lambda)}{\omega(\lambda)} \int_x^b \psi^+(y, \lambda) f(y) dy \\ + \frac{\Delta_{34} \psi^+(x, \lambda)}{\omega(\lambda)} \int_a^{c-} \varphi^-(y, \lambda) f(y) dy + \frac{\Delta_{12} f_1 \varphi^+(x, \lambda)}{\omega(\lambda)}, & \text{for } x \in (c, b] \end{cases} \quad (25)$$

Consequently

$$Y(x, \lambda) = Y_0(x, \lambda) + f_1 \Delta_{12} \frac{\varphi(x, \lambda)}{\omega(\lambda)}. \quad (26)$$

where $G_0(x, \lambda)$ and $Y_0(x, \lambda)$ is the same with (18) and (19) respectively. From the equalities (13) and (18) it follows that

$$(G_0(x, \cdot; \lambda))'_\beta = \frac{\varphi(x, \lambda)}{\omega(\lambda)}. \quad (27)$$

By using (19), (25) and (27) we deduce that

$$\begin{aligned} Y(x, \lambda) &= \Delta_{34} \int_a^{c-} G_0(x, y; \lambda) f(y) dy + \Delta_{12} \int_{c+}^b G_0(x, y; \lambda) f(y) dy \\ &+ f_1 \Delta_{12} (G_0(x, \cdot; \lambda))'_\beta \end{aligned} \quad (28)$$

Consequently, the solution $Y(F, \lambda)$ of the operator equation (20) has the form

$$Y(F, \lambda) = (Y(x, \lambda), (Y(\cdot, \lambda))'_\beta) \quad (29)$$

From (28) and (29) it follows that

$$Y(F, \lambda) = (< G_{x,\lambda}, \overline{F} >_1, (< G_{x,\lambda}, \overline{F} >_1)'_\beta) \quad (30)$$

where under Green's vector $G_{x,\lambda}$ we mean

$$G_{x,\lambda} := (G_0(x, \cdot; \lambda), (G_0(x, \cdot; \lambda))'_\beta) \quad (31)$$

Now, making use (18), (28), (29), (30) and (31) we see that if λ not an eigenvalue of operator A then

$$Y(F, \lambda) \in D(A) \text{ for } F \in H_1, \quad (32)$$

$$Y((\lambda - A)F, \lambda) = F, \text{ for } F \in D(A) \quad (33)$$

and

$$\|Y(F, \lambda)\| \leq |Im\lambda|^{-1} \|F\| \text{ for } F \in H_1, \quad Im\lambda \neq 0. \quad (34)$$

Hence, each nonreal $\lambda \in \mathbb{C}$ is a regular point of an operator A and

$$R(\lambda, A)F = (< G_{x,\lambda}, \overline{F} >_1, (< G_{x,\lambda}, \overline{F} >_1)'_\beta) \text{ for } F \in H_1 \quad (35)$$

Because of (32) and (35)

$$(\lambda - A)D(A) = (\overline{\lambda} - A)D(A) = H_1 \text{ for } Im\lambda \neq 0. \quad (36)$$

Theorem 1. *The Resolvent operator $R(\lambda, A)$ is compact in the Hilbert space H_1 .*

PROOF.

4. Selfadjoint realization of the problem

At first we shall prove the following lemmas.

Lemma 1. *The domain $D(A)$ is dense in H_1 .*

PROOF.

Lemma 2. *The linear operator A is symmetric in the Hilbert space H_1 .*

PROOF. Let $F = (f(x), T'_b(f)), G = (G_1(x), T'_b(f)) \in D(A)$. By partial integration we get

$$\begin{aligned}
\langle AF, G \rangle_1 &= \Delta_{12} \int_a^{c-} (\mathcal{L}f)(x) \overline{g(x)} dx + \Delta_{34} \int_{c+}^b (\mathcal{L}f)(x) \overline{g(x)} dx \\
&+ \frac{\Delta_{34}}{\Delta_0} T_b(f) \overline{T'_b(g)} \\
&= \langle F, AG \rangle_1 + \Delta_{12} W(f, \bar{g}; c-0) - \Delta_{12} W(f, \bar{g}; a) \\
&+ \Delta_{34} W(f, \bar{g}; b) - \Delta_{34} W(f, \bar{g}; c+0) \\
&+ \frac{\Delta_{34}}{\Delta_0} (T'_b(f) \overline{T_b(g)} - T_b(f) \overline{T'_b(g)}). \tag{37}
\end{aligned}$$

From the definition of domain $D(A)$ we see easily that $W(f, \bar{g}; a) = 0$. The direct calculation gives

$$T'_b(f) \overline{T_b(g)} - T_b(f) \overline{T'_b(g)} = -\Delta_0 W(f, \bar{g}; b) \text{ and } W(f, \bar{g}; c-0) = \frac{\Delta_{34}}{\Delta_{12}} W(f, \bar{g}; c+0).$$

Substituting these equalities in (37) we have

$$\langle AF, G \rangle_1 = \langle F, AG \rangle_1 \text{ for every } F, G \in D(A),$$

so the operator A is symmetric in H . The proof is complete.

Remark 2. *By Lemma 2 all eigenvalues of the problem (1) – (5) are real. Therefore it is enough to investigate only real-valued eigenfunctions. Taking in view this fact, we can assume that the eigenfunctions are real-valued.*

Corollary 1. *If λ_n and λ_m are distinct eigenvalues of the problem (1) – (5), then the corresponding eigenfunctions $u_n(x)$ and $u_m(x)$ is orthogonal in the sense of the following equality*

$$\Delta_{12} \int_a^{c-} u(x)v(x)dx + \Delta_{34} \int_{c+}^b u(x)v(x)dx + \frac{\Delta_{34}}{\Delta_0} T'_b(u)T'_b(v) = 0. \tag{38}$$

PROOF. The proof is immediate from the fact that, the eigenelements $(u(x), T'_b(u))$ and $(v(x), T'_b(v))$ of the symmetric linear operator A is orthogonal in the Hilbert space H_1 .

Theorem 2. *The operator A is self-adjoint in H_1 .*

PROOF.

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